AE353: Final Exam

December 18, 2024

You are allowed to use any online or offline notes at your disposal, **but no communication**. You must **write a complete solution, showing the process that you took to reach your answer**.

The entire exam lasts 180 minutes (unless you have been alloted extra time), including all the questions on paper *and* those on PrairieLearn. Computational tools are *not* allowed for the paper element of the exam, but are allowed for PrairieLearn.

Each problem on paper is worth 25 points. Each problem on PrairieLearn is worth 20 points. Consequently, the nominal maximal mark on this exam is 300 points, but it is possible to collect 350. In other words, the "bonus" amount available on this exam is $16.66\dot{6}\%$.

The problems are *not* ranked in order of difficulty.

Problem 1.

Consider a controlled dynamical system with state $q \in \mathbb{R}$ and input $p \in \mathbb{R}$ given by the equation

$$\dddot{q} = 8q^2\ddot{q}p + \sin(p).$$

By adding "dummy states", convert it into a first-order dynamical system described by equations $\dot{w} = f(w, p)$, where w is the extended system state. *Write the definition of w, and carefully write every component of f in terms of w and p.*

Problem 2.

Consider a nonlinear control system with state $w \in \mathbb{R}^2$ and input $p \in \mathbb{R}^2$ given by

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} e^{w_1} + w_2^2 + p_1p_2 + 1 \\ e^{w_1} + w_1w_2 + p_2 \end{pmatrix},$$

with output $z \in \mathbb{R}$ given by $z = e^{w_1}p_1p_2$. Find the equilibrium (w_e, p_e) of this system such that $w_e = 0$. Linearize this system around this equilibrium. Express the linearized system in the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, and make sure to write down the relationships between x and w, u and p, and y and z.

Problem 3.

Consider the linear control system

$$\dot{x}(t) = \begin{pmatrix} 2 & 7 \\ -5 & 5 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 5 \end{pmatrix} u(t),$$

with $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$. Let

$$u(t) = \begin{pmatrix} 1 & -1 \end{pmatrix} x(t)$$

for all t. Find $x(5)$.

Optional suggestion: determine x_2 before x_1.

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Problem 4.

Consider the linear control system

$$\dot{x}(t) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u(t).$$

(a) Determine whether this system is controllable.

(b) Determine whether this system is stabilizable.

(c) Find a control signal u which ensures $x_3(t) \to 0$ regardless of $x_3(0)$.

Optional suggestion for (c): just consider the dynamics of x_3 and design u accordingly.

Problem 5.

Consider a controlled dynamical system with state $z \in \mathbb{R}$ and input $\tau \in \mathbb{R}$ described by

$$\dot{z} = e^{z\tau}.$$

Let $w = e^z$. Describe the dynamics of w, i.e., find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\dot{w} = f(w, \tau)$.

Problem 6.

Consider matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Show that M can be written as a sum of a *symmetric matrix* P and an *antisymmetric matrix* Q, i.e., that $M = P + Q$, where $P = P^T$ and $Q = -Q^T$.

Problem 7.

Consider a linear control system given by $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 3}$ and $A^T B = 0$. Consider the problem of finding a control signal u which minimizes

$$\int_0^1 \|\dot{x}(t)\|^2 dt.$$

Convert it (you do not have to solve the problem!) into the LQR problem of minimizing

$$\int_0^1 (x(t))^T Q x(t) + (u(t))^T R u(t) dt,$$

i.e., find positive semidefinite matrices Q and R such that the two problems match. If you wish, you can express matrices Q and R in terms of A and B.

(Any matrix M of the form $M = N^T N$ is automatically positive semidefinite; you do not have to verify this fact.)

Problem 8.

Consider a single-state linear control system given by

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

Assume that $B \neq 0$ and $C \neq 0$. Show that, regardless of B and C, this system is detectable, observable, controllable and stabilizable.

Emphasis: the system is single-state, not necessarily single-input or single-output!

Problem 9.

Consider a linear control system given by

$$\dot{x} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 18 \end{pmatrix} u, \quad y = \begin{pmatrix} 27 & 0 \end{pmatrix} x.$$

As we did in class, construct an asymptotically correct observer $\hat{x}$, i.e., a function $\hat{x}$ such that

$$\lim_{t \to +\infty} \hat{x}(t) - x(t) = 0.$$

You can describe $\hat{x}$ through an ordinary differential equation *and* an initial value.

Problem 10.

Consider a single-input, single-state linear control system given by $\dot{x} = 2x - u$, with $x(0) = 1$. Show that there does not exist a minimum of

$$\int_0^1 x^2(t) - u^2(t) dt,$$

i.e., that one can choose u such that the integral above is negative and arbitrarily large in magnitude.