AE353: Midterm

November 13, 2024

You are allowed to use any online or offline notes at your disposal, **but no communication and computational tools**. You must **write a complete solution, showing the process that you took to reach your answer**. The problems are **not** sorted by difficulty.

Problem 1. (25 points)

Consider matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $N = (MM^T)^{-1}$. Show that N is symmetric, i.e., that $N = N^T$.

(I suggest working smart, not hard.)

Problem 2. (25 points)

This problem had a typo that allowed it to be understood in multiple ways. This document contains the problem and the solution that were originally intended, but a correct solution to any interpretation is equally acceptable.

Consider a linear control system given by $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 2}$. Consider the problem of finding a control signal u which minimizes

$$\int_0^1 (x_1(t) + x_2(t))^2 + (x_1(t) - x_2(t))^2 + \|u(t)\|^2 dt.$$

Convert it (you do not have to solve the problem!) into the LQR problem of minimizing

$$\int_0^1 (x(t))^T Q x(t) + (u(t))^T R u(t) dt,$$

i.e., find positive semidefinite matrices Q and R such that the two problems match.

Problem 3. (25 points)

Consider a two-state linear control system given by

$$\dot{x} = Ax + Bu, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

Assume that all elements of A are non-zero. Show that this system is detectable and observable.

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Problem 4. (25 points)

Consider a linear control system given by

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

Assuming that the controller can only read output values $y(t)$, but not full $x(t)$ directly, construct a control signal u which ensures $\lim_{t\to+\infty} x(t) = 0$.

You do not need to write $u(t)$ explicitly for all t, but, if you want, you can write it as an output of an observer system.

Bonus Problem. (25 points)

Consider a single-input, single-state linear control system given by $\dot{x} = x + u$, with $x(0) = 1$. Show that there does not exist a minimum of

$$\int_0^1 x^2(t) - u^2(t)dt,$$

i.e., that one can choose u such that the integral above is negative and arbitrarily large in magnitude.